Path Optimization Through A Bounded Region

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Abstract

The shortest path between two points is a straight line. However, finding the optimum path from one point to another with the introduction of a given boundary condition on the domain between the two points does not possess such a trivial solution. The boundary condition is expressed in the form of a twice-differentiable function and serves as an upper or lower bound for the desired path depending on characteristics of the boundary function such as concavity and extrema behaviors on the domain of interest. The obtained path that minimizes arc length without violating the boundary function is deemed the Rubber Band Solution. This solution type is named due to its similarities to the path that a rubber band creates when it is stretched around an obstacle by minimizing the potential energy from the elastic forces in the band. Since we seek an analytic solution, we first consider using boundary arcs that are circles. We then generalize to boundary arcs that can be described by any differentiable function by finding the circle of curvature at the the local max/min of the function on the desired domain. The case where the boundary arcs are circles serves as an approximation for the optimized path around any of the more general boundary functions.

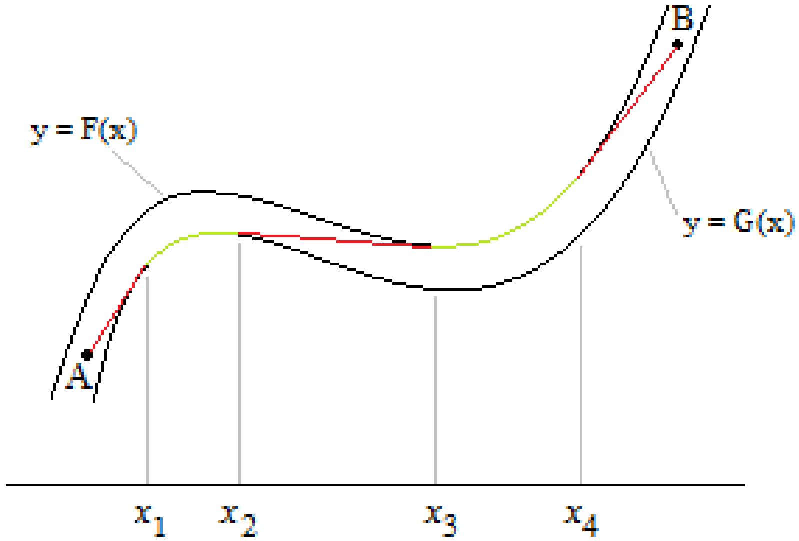
Creating the Region and Defining Optimization

Commonly it is the objective of individuals to get from one point to another in a desired manner. For example, it is typically the objective of a GPS system to inform a driver of the quickest, or in other words, the time optimized path to get from an initial location to a desired destination. Clearly, the GPS unit is calculating route times for numerous different paths that the driver could take. After potential paths have been eliminated and trip times have been calculated, the unit returns the route that will take the least amount of time for the driver to reach his/her destination. For the problem in this paper, we assume that to get from an initial location to a destination, there is only a single route and the traveler is the only driver on the road. The route is a multilane highway that connects the initial and final locations and is not necessarily straight from start to finish. Assume also that the driver never deviates from the speed limit from the start to the finish of the trip. Therefore, with this setup, an optimized path must be a path that allows the driver to minimize the distance traveled. The driver is restricted to the multilane highway that connects their start point to their destination. Thus, the optimized path must be some path within the boundaries of the multilane road that allows for a minimum distance relative to other possible routes within the restricted road parameters.

In two-space, it is our desire to find the optimized path between two points such that the path must not cross over a set of bounds defining the region modeling the multilane highway. To this end, we will first assume that the bounds of this highway can be modeled by two twice-differentiable (likely offset) functions; one defining a lower bound and the other defining the upper bound. Thus, the region of optimization will be between (and including) these boundary curves. Let A denote the initial point and B denote the destination point, and suppose that these points are within the region of optimization. The optimal (shortest) path from A to B is one that must stay within the region of optimization, and it is allowed to travel along any of the boundary curves. Moreover, this optimal path Γ will solve the problem:

minimize arc length of γ, where γ is any path from A to B in the region

We show later that Γ is a piecewise function with a continuous first derivative in which some portions are straight line segments tangent to one (or both) of the boundary curves at certain points, and other portions follow a boundary curve. The following illustration shows an example of what we would expect Γ would look like for a highway bounded by the offset curves pictured. The portions highlighted in green show where Γ is allowed to trace either of the boundary curves, and the portions in red are where Γ is in the interior of the region.



An important thing to note in the illustration are the 4 points with x-coordinates labeled x_1, x_2, x_3, and x_4. These points in this example will define the points on the optimal path Γ where the portions defined by segments (red) will be tangent to a boundary curve.

In order to approach this problem in a manner that will yield analytic solutions, it is necessary to use a very specific kind of boundary function. Although any continuous function can be a boundary function in the most general case, it is difficult to find a closed solution defining specific optimized paths that can be expressed as piecewise functions. In particular, it is currently unknown if there exists an analytic expression for finding the points at which the segment portions of Γ would be tangent to a given boundary curve defined by $y = f(x)$. To deal with this difficulty, we first consider using a circle as one bounding curve. In this case, we are able to obtain a closed formula for computing the points of tangency with the portions of Γ that are line segments. Then we generalize this to the case in which one of the boundaries is modeled by a twice-differentiable function $y = f(x)$ by considering circles and radii of curvature. In particular, we compute the circles of curvature of $y = f(x)$ tangent to a local min or max and approximate the points where segment portions of Γ will be tangent to $y = f(x)$ using the analysis in the case where the boundary is a circle.

The Simplest Case - A Circle Centered at $(0, 0)$

We first consider an obstacle circle centered at the origin and endpoints, A and B, lying on the x-axis. Later, we will consider the more general case where A is either in quadrant II or III, and B is either in quadrant I or IV. In general, the equation of the circle of radius r centered at (x_0, y_0) is

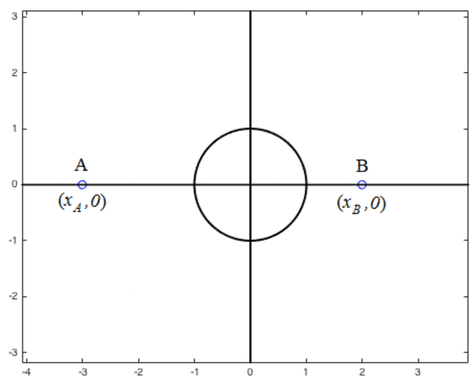
$$(x - x_0)^2 + (y - y_0)^2 = r^2. \tag{1}$$

Moreover, we denote the coordinates of points A and B as follows:

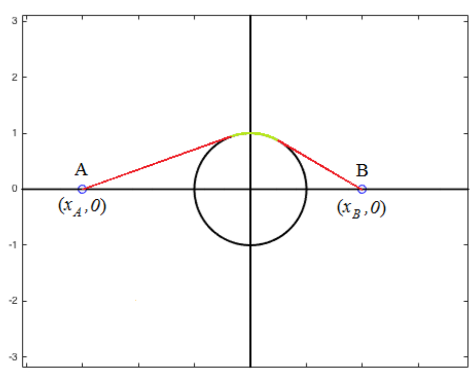
$$A = (x_A, y_A), \quad B = (x_B, y_B). \tag{2}$$

In our current case, we have $x_0 = y_0 = 0$, and $y_A = y_B = 0$. Therefore, (1) and (2) imply that we are solving the problem:

Find the shortest path from $(x_A, 0)$ to $(x_B, 0)$ around the obstacle $x^2 + y^2 = r^2$.

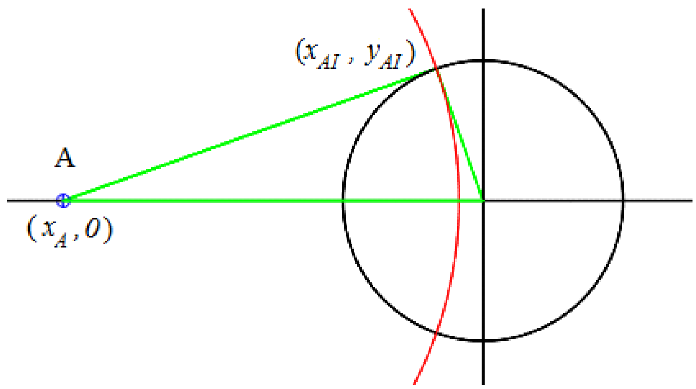


The proposed solution will follow a path from point A along a straight line to the tangential point on the boundary arc then follow the boundary arc until a point where a straight line to point B is tangential to the arc and at that time will exit the arc and follow the straight path to B.



Our next goal is to find the coordinates of the point where the line through A is tangent to the circle $x^2 + y^2 = r^2$. Let (x_{AI}, y_{AI}) denote this point of tangency.

Thus, the segment from point A to (x_{AI}, y_{AI}) will give the first section of the optimal path from point A to B. Let R_A denote the distance from point A to the center of the obstacle circle (in our current case, $R_A = -x_A$). The following Theorem establishes the location of the point (x_{AI}, y_{AI}).



Theorem 1: $(x_{AI}, y_{AI}) = \left(\dfrac{-r^2}{R_A}, r\sqrt{1 - (\frac{r}{R_A})^2} \right)$

Proof of Theorem 1: According to the Tangent Line Theorem in [1], the segment from point A to (x_{AI}, y_{AI}) is perpendicular to the segment from $(0, 0)$, to (x_{AI}, y_{AI}). Therefore, the triangle formed (in green) in the above illustration is a right triangle. Let D denote the distance from point A to (x_{AI}, y_{AI}). According the the Pythagorean Theorem,

$$
\begin{aligned}
r^2 + D^2 &= R_A^2 \\
\Rightarrow D^2 &= R_A^2 - r^2 \qquad (3)
\end{aligned}
$$

Next, we construct the circle centered at point A with radius D. The equation of this circle is

$$(x - x_A)^2 + y^2 = D^2 \qquad (4)$$

Note that circle (4) will intersect the obstacle circle, $x^2 + y^2 = r^2$, in two places. We will choose the intersection point on the upper half of the plane. Solving $x^2 + y^2 = r^2$ for y, we obtain the equation for the top semicircle $y = \sqrt{r^2 - x^2}$. Similarly, solving (4) for y yields the top semicircle $y = \sqrt{D^2 - (x - x_A)^2}$. The x-coordinate of the point of tangency, x_{AI}, will be the same as the x-coordinate of the point of intersection between semicircles $y = \sqrt{r^2 - x^2}$ and

$y = \sqrt{D^2 - (x - x_A)^2}$. This implies that x_{AI} satisfies the following:

$$\begin{aligned}
\sqrt{r^2 - x_{AI}^2} &= \sqrt{D^2 - (x_{AI} - x_A)^2} \\
&\Rightarrow r^2 - x_{AI}^2 = D^2 - (x_{AI} - x_A)^2 \\
&\Rightarrow r^2 - x_{AI}^2 = R_A^2 - r^2 - (x_{AI} - x_A)^2 \text{ by (3)} \\
&\Rightarrow r^2 - x_{AI}^2 = R_A^2 - r^2 - (x_{AI} - (-R_A))^2 \text{ since } R_A = -x_A \text{ in this case} \\
&\Rightarrow 2r^2 - x_{AI}^2 = R_A^2 - (x_{AI} + R_A)^2 \\
&\Rightarrow 2r^2 - x_{AI}^2 = R_A^2 - (x_{AI}^2 + 2R_A x_{AI} + R_A^2) \\
&\Rightarrow 2r^2 - x_{AI}^2 = R_A^2 - x_{AI}^2 - 2R_A x_{AI} - R_A^2 \\
&\Rightarrow 2r^2 = -2R_A x_{AI} \\
&\Rightarrow \frac{-r^2}{R_A} = x_{AI}. \qquad (5)
\end{aligned}$$

Since (x_{AI}, y_{AI}) lies on the obstacle circle, $y_{AI} = \sqrt{r^2 - x_{AI}^2}$ must be satisfied. Substitution of (5) into this equation leads to $y_{AI} = r\sqrt{1 - (\frac{r}{R_A})^2}$. This completes the proof.

Now that we have the point of tangency established, we can easily find the equation of the line through points A and (x_{AI}, y_{AI}). The slope of this line is given by

$$m = \frac{(y_{AI} - y_A)}{(x_{AI} - x_A)}$$

Therefore, the equation of the line defining the first segment along the shortest path is given by

$$y = \frac{y_{AI} - y_A}{x_{AI} - x_A}(x - x_A) + y_A \qquad (6)$$

Next, the line through B tangent to the obstacle circle needs to be established. We can employ the same method used earlier; define R_B to be the distance from point B to the center of the obstacle circle (which is $(0, 0)$ in this case, and let (x_{BI}, y_{BI}) be the point of tangency. Using an analogue to Theorem 1, one obtains

$$(x_{BI}, y_{BI}) = \left(\frac{r^2}{R_B}, r\sqrt{1 - \left(\frac{r}{R_B}\right)^2}\right) \qquad (7)$$

Then (similar to (6)) one finds that the equation of the line defining the last segment along the shortest path is given by

$$y = \frac{y_{BI} - y_B}{x_{BI} - x_B}(x - x_B) + y_B \qquad (8)$$

Summarizing all of the results in the simplest case, we have the following result:

Theorem 2 - The Rubber Band Solution for the Simplest Case: Suppose $r > 0$, and consider the obstacle circle $x^2 + y^2 = r^2$. Let $A = (x_A, 0)$, $B = (x_B, 0)$, where $x_A < -r$ and $x_B > r$. Then the shortest path (a.k.a. the Rubber Band Solution) from A to B around the obstacle circle is defined by the piecewise function

$$R(x) = \begin{cases} y = [y_{AI}/(x_{AI} - x_A)](x - x_A) & \text{if } x \in [x_A, x_{AI}] \\ y = \sqrt{r^2 - x^2} & \text{if } x \in [x_{AI}, x_{BI}] \\ y = [y_{BI}/(x_{BI} - x_B)](x - x_B) & \text{if } x \in [x_{BI}, x_B] \end{cases},$$

where $x_{AI} = \dfrac{r^2}{x_A}$ and $x_{BI} = \dfrac{r^2}{x_B}$.

Remark: This follows from (5), (6), (7), and (8) because $y_A = y_B = 0$, $R_A = -x_A$, and $R_B = x_B$.

The General Case - Circle Centered at (x_0, y_0)

Our next step is to extend this process to a more general case with a circle centered at any point (x_0, y_0). In this case, the equation for the obstacle arc is given in (1), namely.

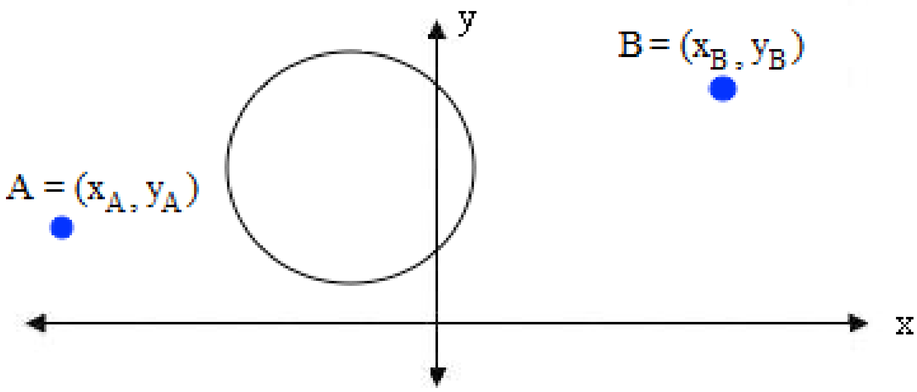
$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Furthermore, the endpoints A and B will also be generalized. We will no longer assume that these points lie on the line defining a diameter of the obstacle circle. We will, however, still assume that the line $\overleftrightarrow{AB}$ intersects the obstacle circle in two points. In other words, the x and y values for the endpoints are such that a straight-line solution is not obtainable. If $A = (x_A, y_A)$ and $B = (x_B, y_B)$, then the following must be satisfied:

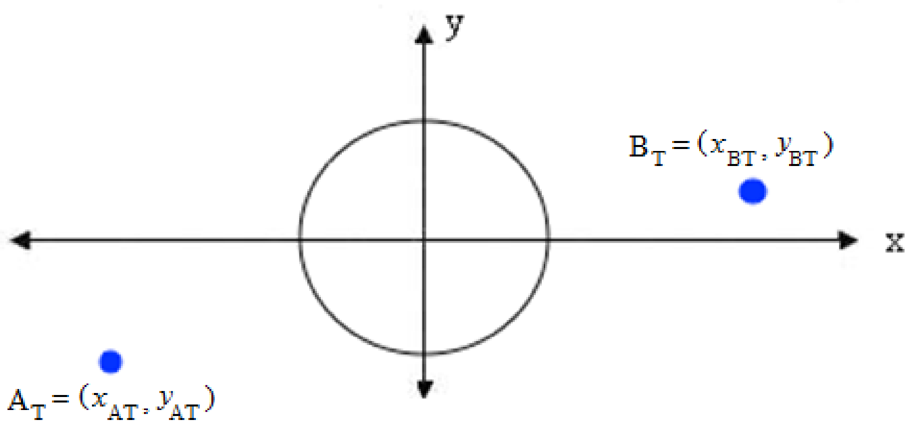
$$\sqrt{(x_{A,B} - x_0)^2 + (y_{A,B} - y_0)^2} > |r|. \tag{9}$$

Thus, in this case we are interested in the following problem (as given by the criteria in (1) and (2)):

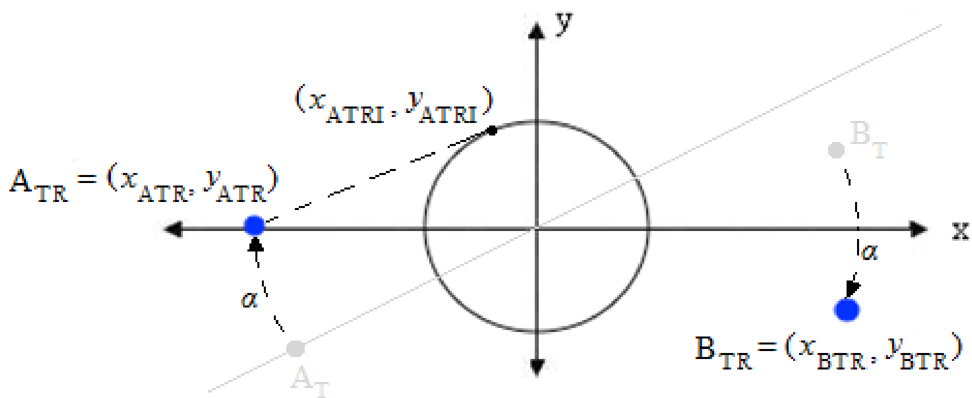
Find the shortest path from (x_A, y_A) to (x_B, y_B) around the obstacle $(x - x_0)^2 + (y - y_0)^2 = r^2$.



We first translate the boundary circle back to the origin, and we shift the points A and B by the same amount (horizontally by x_0 units and vertically by y_0 units). Let A_T and B_T denote the image of points A and B, respectively, after the translation.



After this is done, the points A_T and B_T may not lie on the x-axis (as points A and B did in the first case). Thus we will also rotate the circle along with point A_T by an appropriate amount so that the new point, call it A_{TR}, lies on the negative x-axis (as A did in the first case). Then we apply the results from the first case to finding the line from A_{TR} tangent to the translated circle. The same process is applied to point B_{TR}, the image of B_T after an appropriate rotation maps B_{TR} onto the positive x-axis.



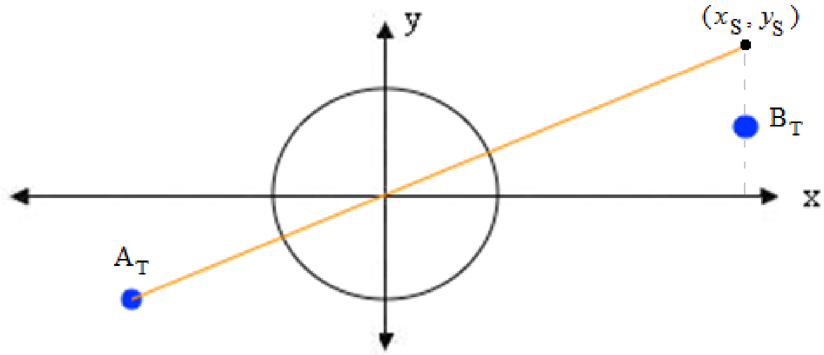
With the process outlined, let's first establish the images of points A and B under the translation:

$$A_T = (x_A - x_0, y_A - y_0) = (x_{AT}, y_{AT}) \tag{10}$$

$$B_T = (x_B - x_0, y_B - y_0) = (x_{BT}, y_{BT}) \tag{11}$$

Note that it is realistic to maintain endpoints on opposite sides of the vertical axis through the obstacle circle, the y-axis for the translated obstacle circle, as well as y values inside the range of the obstacle arc.

Due to the translation, the y value of the endpoints will most likely be non-zero. Therefore, we must introduce a few new steps in the process to deal with this dilemma. Our first goal is to decide what side of the obstacle circle is the best to create a path around. It is natural to think that depending on the placement of the endpoints relative to the center of the circle and one another, that connecting the points around the top of the circle might be shorter than connecting them around the bottom, or vise versa. To test which path direction will yield the true optimum arc, we will test the placement of one point relative to the other and the center of the circle. To do this, we construct a line of symmetry passing through the center of the obstacle circle. This line will pass through point A and the origin.



Thus the equation of this line of symmetry (indicated in orange in the figure) is as follows:

$$\begin{aligned} y - (y_A - y_0) &= \frac{y_A - y_0}{x_A - x_0}[x - (x_A - x_0)] \\ &\Rightarrow y - y_{AT} = \frac{y_{AT}}{x_{AT}}(x - x_{AT}) \\ &\Rightarrow y = \frac{y_{AT}}{x_{AT}}(x - x_{AT}) + y_{AT} \\ &\Rightarrow f_S(x) = \frac{y_{AT}}{x_{AT}}(x - x_{AT}) + y_{AT} \qquad \Rightarrow f_S(x) = \frac{y_{AT}}{x_{AT}} \end{aligned} \tag{12}$$

In the above illustration, the point (x_S, y_S) is defined by $(x_{BT}, f_S(x_{BT}))$. Testing point B with respect to this line will enlighten us to the direction we must go around the obstacle. If point B lies above the line, or

$$y_{BT} \geq y_S,$$

then a path around the top of the obstacle circle will be the shortest route. Similarly, if point B lies below the line of symmetry, or

$$y_{BT} \leq y_S,$$

the path direction must be below the obstacle circle. If B lies on the line of symmetry, both routes will yield equal arc lengths so either one is a viable path.

Going over both cases, a path over the top and a path around the bottom, is repetitive so illustrating one of the two will be sufficient for showing the necessary techniques to deal with the added generality. The top route will be illustrated here and the bottom route will be included in the full process illustration in Appendix A.

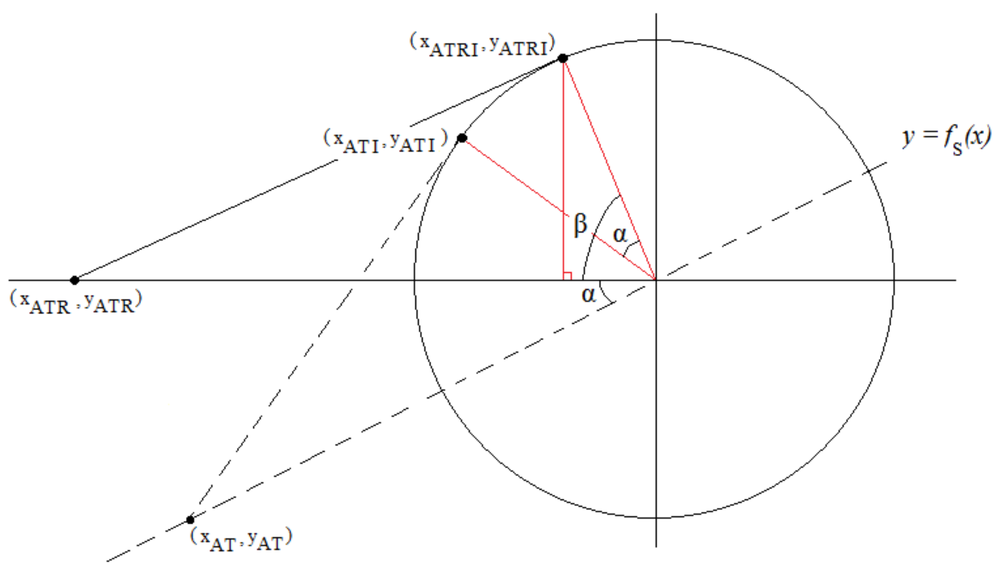
The first step to creating the solution by finding the tangential points to the obstacle arc with off axis endpoints is to place the endpoints on the axis; as was the case in the previous section. To do this, a rotation of an endpoint will be made, and the obstacle circle will be rotated about the origin by the same amount to create a situation identical to the simplest case. Note that rotations of the entire system, like the translations, will not affect the accuracy of the solution due to the isotropy of the system in two-space. Rotating the endpoint to the horizontal axis coupled with the rotation of the obstacle circle is a valid step as long as the entire system is rotated and then rotated back to its original position to create the final solution. The rotation of the A endpoint to the x axis creates the following point that will then be used to find the tangent point on the obstacle circle at which point both points will be rotated in an equal and opposite manner to the rotation of A to the axis initially.

$$A_{TR} = (x_{ATR}, y_{ATR}) = \left(-\sqrt{x_{AT}^2 + y_{AT}^2}, 0\right) \tag{13}$$

The tangent point on the obstacle arc follows identically from Theorem 1, the proof of which has been established. Using the most inclusive notation, the tangent point (x_{ATRI}, y_{ATRI}) is:

$$(x_{ATRI}, y_{ATRI}) = \left(\frac{-r^2}{R_A}, r\sqrt{1 - \left(\frac{r}{R_A}\right)^2}\right) \tag{14}$$

In (14), $R_A = \sqrt{x_{AT}^2 + y_{AT}^2}$. Now that the tangent point and the A endpoint are defined in a rotated state, it is necessary to rotate each of them to their correct positions. Clearly, A returns to (x_{AT}, y_{AT}) while (x_{ATRI}, y_{ATRI}) undergoes an equal rotation to a currently unknown position, namely (x_{ATI}, y_{ATI}) (See illustration below).



The following Theorem establishes the location of the point (x_{ATI}, y_{ATI}) if we choose to go over the top of the obstacle.

Theorem 3: $(x_{ATI}, y_{ATI}) = (r\cos\theta_A, r\sin\theta_A)$, where

$$\theta_A = \pi + \alpha + \beta,\ \tan(\alpha) = \frac{y_{AT}}{x_{AT}},\ \text{and}\ \tan(\beta) = \frac{y_{ATRI}}{x_{ATRI}}.$$

Proof of Theorem 3: To consider the proper angle θ_A that we will rotate by, we shall examine two cases. In each case, we define angles α and β as follows: α is the angle between (x_{ATRI}, y_{ATRI}) and (x_{ATI}, y_{ATI}) and β is the interior angle of the right triangle formed by $(x_{ATRI}, 0)$, (x_{ATRI}, y_{ATRI}), and the origin. α and β are clear from the illustration on the previous page, but the illustration shown is valid in the case when $y_{AT} < 0$.

Case I: Suppose $y_{AT} < 0$. Since $x_{AT} < 0$, then

$$\alpha = \tan^{-1}\left(\left|\frac{y_{AT}}{x_{AT}}\right|\right) = \tan^{-1}\left(\frac{y_{AT}}{x_{AT}}\right) \tag{15}$$

Since we are choosing to go over the top of the obstacle, $x_{ATRI} < 0$ and $y_{ATRI} > 0$. Thus

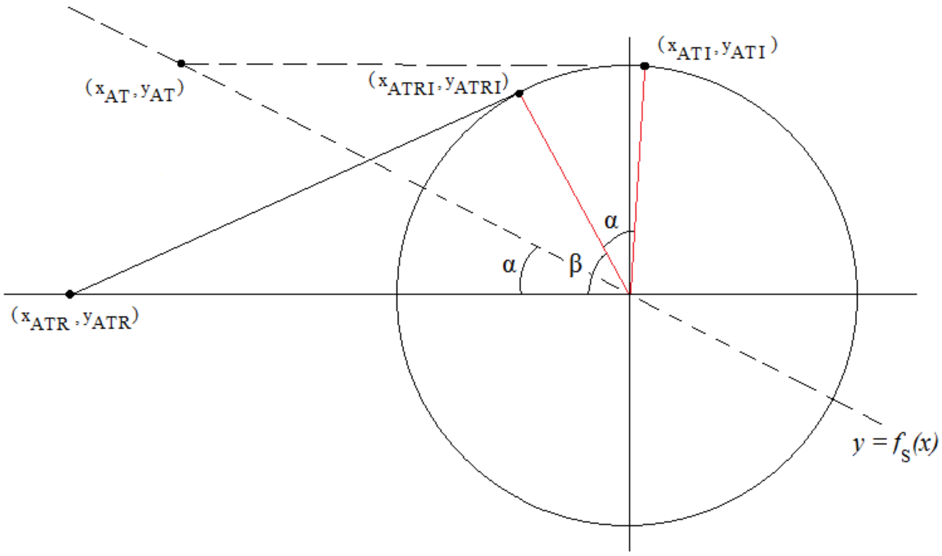
$$\alpha = \tan^{-1}\left(\left|\frac{y_{ATRI}}{x_{ATRI}}\right|\right) = -\tan^{-1}\left(\frac{y_{ATRI}}{x_{ATRI}}\right) \tag{16}$$

Therefore it follows from (15) and (16) (as can be seen in the previous illustration)

$$\theta_A = \pi - (\beta - \alpha) = \pi + \tan^{-1}\left(\frac{y_{AT}}{x_{AT}}\right) + \tan^{-1}\left(\frac{y_{ATRI}}{x_{ATRI}}\right)$$

Case II: Suppose $y_{AT} > 0$. Since $x_{AT} < 0$, then

$$\alpha = \tan^{-1}\left(\left|\frac{y_{AT}}{x_{AT}}\right|\right) = -\tan^{-1}\left(\frac{y_{AT}}{x_{AT}}\right) \tag{17}$$



Since we are choosing to go over the top of the obstacle, $x_{ATRI} < 0$ and $y_{ATRI} > 0$. Thus

$$\alpha = \tan^{-1}\left(\left|\frac{y_{ATRI}}{x_{ATRI}}\right|\right) = -\tan^{-1}\left(\frac{y_{ATRI}}{x_{ATRI}}\right) \tag{18}$$

Therefore it follows from (17) and (18) (as can be seen in the previous illustration)

$$\theta_A = \pi - (\alpha + \beta) = \pi + \tan^{-1}\left(\frac{y_{AT}}{x_{AT}}\right) + \tan^{-1}\left(\frac{y_{ATRI}}{x_{ATRI}}\right)$$

This completes the proof for Theorem 3 based on endpoint A and a path over the top of the obstacle arc.

Like before, the process is the same for the B segment of the path. Once the tangent point value is defined for the B segment, all that is left in the process is to translate the entire system back to its original center and create the tangent line pieces of the solution using the newly retranslated endpoints and tangent points.

The completion of this process yields the following result:

Theorem 4 - The Rubber Band Solution for the General Case: Suppose $r > 0$, and consider the obstacle circle $(x - x_0)^2 + (y - y_0)^2 = r^2$. Let $A = (x_A, y_A)$, $B = (x_B, y_B)$, where $\sqrt{(x_A - x_0)^2 + (y_A - y_0)^2} > |r|$ and $\sqrt{(x_B - x_0)^2 + (y_B - y_0)^2} > |r|$. Then the shortest path (a.k.a. the Rubber

Band Solution) from A to B around the obstacle circle is defined by the piecewise function

$$R(x) = \begin{cases} y = \dfrac{(y_{AI} - y_A)}{(x_{AI} - x_A)}(x - x_A) + y_A & \text{if } x \in [x_A, x_{AI}] \\ y = \pm\sqrt{r^2 - (x - x_0)^2} + y_0 & \text{if } x \in [x_{AI}, x_{BI}] \\ y = \dfrac{(y_{BI} - y_B)}{(x_{BI} - x_B)}(x - x_B) + y_B & \text{if } x \in [x_{BI}, x_B] \end{cases},$$

Remark: In the second line of the definition of R, the $+$ in $\pm$ is chosen if the shortest path is over the circle, and the $-$ is chosen if the shortest path is under the circle.

Added Malleability

The above demonstration of an optimum path is the most general solution for a given circle as an obstacle curve, but how can this be extended to a more general set of boundary functions? For this extension, an exact analytic solution will not be our focus. Instead, a general solution using an approximation method based on the analytic solution from the obstacle circle will be employed to help incorporate boundaries that are not circular arcs. To do this, the radius of curvature for a function will be used to create a circle imbedded in the boundary function that can then be used to approximate the optimum path around the given boundary function. Before any illustrations of the beginnings of this process, it is helpful to look at the general expression for radius of curvature. Given a function $f(x)$,

$$R_c = \frac{[1 + f'(x)^2]^{\frac{3}{2}}}{f''(x)} \tag{19}$$

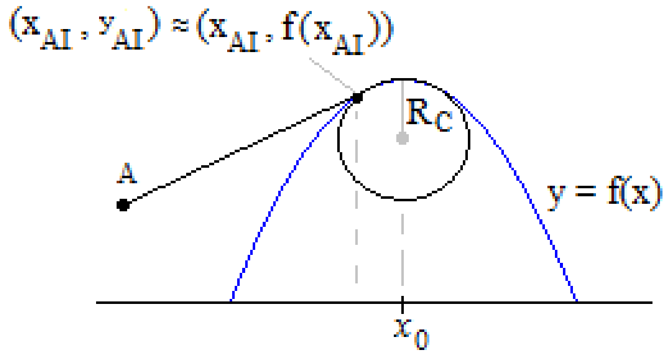
Note that the boundary function, $f(x)$ must be twice differentiable in order to satisfy the above equation. Another thing to note: In (19), the value of R_c is allowed to be negative. In fact, $R_c < 0$ for all x in which f is concave down.

It is our concern to know where to construct the circle using the radius of curvature. To do this, the first derivative of the function must be evaluated and set equal to zero to find the x-value where a maximum or minimum occurs on the expressed domain depending on the concavity of $f(x)$ about the relative extrema. In other words, the value x_0 is the x-value satisfying the following equation

$$f'(x) = 0 : x \in [x_A, x_B] \tag{20}$$

From there, the center coordinates and the radius of the needed obstacle circle can easily be expressed as follows using the known values.

$$(x_0, y_0) = (x_0, f(x_0) + R_c) \tag{21}$$



Note that x_0 is the x-value corresponding to a critical point of $f(x)$ on the interval.

$$(x - x_0)^2 + (y - (f(x_0) + R_c))^2 = R_c^2 \tag{22}$$

It is necessary to know that it might be the case where the line of symmetry condition conflicts with the boundary type of the equation. For example, if the boundary function is a lower bound but the symmetry relationship implies a path underneath the circle, it is a violation of the bound to create this path. Therefore, the path across the top of the obstacle circle would need to be used in order to create a path that is sensible for the boundary function. More concretely, the answer to the needed path for this set up comes from a concavity test about the relative extrema. Knowing this allows for the calculations as explained in the previous section to be carried out and the creation of an approximate optimum path.

Theorem 5 - The Rubber Band Solution Approximation for a General Boundary Function: Let $F(x)$ be a boundary function and allow $A = (x_A, y_A)$, $B = (x_B, y_B)$, such that a straight line from A to B intersects $F(x)$ twice. Then the shortest path (a.k.a. the Rubber Band Solution) from A to B in the bounded region is defined by the piecewise function

$$R(x) = \begin{cases} y = \dfrac{(y_{AI} - y_A)}{(x_{AI} - x_A)}(x - x_A) + y_A & \text{if } x \in [x_A, x_{AI}] \\ y = \sqrt{r^2 - (x - x_0)^2} + y_0 & \text{if } x \in [x_{AI}, x_{BI}] \\ y = \dfrac{(y_{BI} - y_B)}{(x_{BI} - x_B)}(x - x_B) & \text{if } x \in [x_{BI}, x_B] \end{cases},$$

where $(x_{AI}, y_{AI}) = \left(-\dfrac{R_C^2}{R_A}, \pm\sqrt{R_C^2 - (x_{AI})^2} + (f(x_0) \pm R_C)\right)$
and $(x_{BI}, y_{BI}) = \left(\dfrac{R_C^2}{R_B}, \pm\sqrt{R_C^2 - (x_{BI})^2} + (f(x_0) \pm R_C)\right)$

Many standard functions in the real plane would create circles of curvature that work well as an approximation for optimization. Generally, any function with a defined and nonzero second derivative on a concerned interval would fit this approximation technique. More specifically, any function with at least one local extrema to create an obstacle circle about allows for the most precise application of the above procedure.

The Importance of Proof

It may be enlightening for the reader to have more background on the inspiration for the above method. Imagine setting a sphere with a flat bottom on a tabletop in front of you. I like to think of a Magic 8 Ball sitting looking glass down on the table. Now take a lengthy rubber band (about the length of the circumference of the sphere) and stretch it between your two index fingers. You can imagine the pull of the rubber band on both of your fingers as it tends back to its original unstretched state. Now, with rubber band extended between your fingers, lower your hands so that each of your two index fingers hovers above the table on either side of the sphere. The rubber band is now going from one of your fingers, over the top of the sphere, and then connecting to your other finger. Like before, the rubber band is tending back to its unstretched state. Physically, the elastic nature of the rubber band is tending back to an equilibrium state with a significantly lower potential energy between the atoms in the band. The effects of this are the downward force on the sphere and the inward force on your fingers. You can imagine the shape of the rubber band matching that of the piecewise function illustrated before; going from one finger directly to the tangent point on the surface of the sphere and then following the sphere's surface across to the next tangent point and finally to your other finger. It is senseless to think that the rubber band may slouch lower than the tangent point or somehow rise above it to minimize its internal potential energy. Therefore, we can know with a significant level of confidence based on this natural illustration that the path we have defined previously is indeed the optimum path.

However to know with complete confidence, we demonstrate with logical proof.

Proof:
Let $\epsilon > 0$ and $\delta > 0$.
Let $f(x, y) = r^2$ such that:

$$x^2 + y^2 = r^2$$
$$\Rightarrow y = \sqrt{r^2 - x^2}$$

Due to the symmetry of the system about the x and y axis and the homogeneity of two-space, we will use the positive portion of the arc centered at $(0, 0)$ without loss of generality. Thus,

$$f(x) = \sqrt{r^2 - x^2} \tag{23}$$

Allow the point $A = (x_A, y_A) : \sqrt{x_A^2 + y_A^2} > |r|$ and $x_A < 0$
Let $T = (x_T, y_T) : x_T = \dfrac{-r^2}{\sqrt{x_A^2 + y_A^2}} = \dfrac{-r^2}{R_A}$ and $y_T = f(x_T)$.

Let $R(x)$ and $g(x)$ be functions on the interval $[x_A, x_M]$ where $x_M = 0$ such that

$$f(x_M) = R(x_M) = g(x_M) \tag{24}$$

and

$$f(x_A) = R(x_A) = g(x_A) \tag{25}$$

Now we proceed by cases.
Case I:

$$x_\epsilon = x_T - \epsilon \tag{26}$$

$$f(x_\epsilon) = g(x_\epsilon) < R(x_\epsilon) \tag{27}$$

$$R(x_i) > g(x_i) \forall x_i \in (x_A, x_T) \tag{28}$$

$$f(x_T) = R(x_T) = g(x_T) \tag{29}$$

$$f(x_i) = R(x_i) = g(x_i) \forall x \in [x_T, x_M] \tag{30}$$

See that the arc length for $R(x)$ on $[x_A, x_T]$ is

$$S(R(x)) = \sqrt{(f(x_T) - y_A)^2 + (x_T - x_A)^2}$$

Now we look at the arc length of $g(x)$ on $[x_A, x_T]$

$$S(g(x)) = \sqrt{(f(x_\epsilon) - y_A)^2 + (x_\epsilon - x_A)^2} + r\theta$$

where θ is the angle between $g(x_\epsilon)$ and $g(x_T)$. Thus $|\theta| > 0$ due to $x_\epsilon < x_T$
Looking at the following inequalities,

$$S(R(x)) < \sqrt{(f(x_\epsilon) - y_A)^2 + (x_\epsilon - x_A)^2} + \sqrt{(f(x_\epsilon) - f(x_T))^2 + (x_\epsilon - x_T)^2}$$

and

$$\sqrt{(f(x_\epsilon) - y_A)^2 + (x_\epsilon - x_A)^2} + \sqrt{(f(x_\epsilon) - f(x_T))^2 + (x_\epsilon - x_T)^2} < S(g(x))$$

implies justly that

$$S(R(x)) < S(g(x)) : x \in [x_A, x_T] \tag{31}$$

Note that since for this case, $R(x) = g(x) \forall x \in [x_T, x_M]$, this above inequality regarding the arc lengths on the domain $[x_A, x_T]$ is sufficient for the entire case. Therefore,

$$S(R(x)) < S(g(x)) : x \in [x_A, x_M] \tag{32}$$

Case II:

$$x_\epsilon = x_T + \epsilon \tag{33}$$

$$f(x_\epsilon) = R(x_\epsilon) = g(x_\epsilon) \tag{34}$$

$$f(x_T) = R(x_T) < g(x_T) \tag{35}$$

$$R(x_i) < g(x_i) \forall x_i \in (x_A, x_\epsilon) \tag{36}$$

$$f(x_i) = R(x_i) = g(x_i) \forall x \in [x_T, x_M] \tag{37}$$

In order to show that the arc length of $g(x)$, $S(g(x))$, is greater than $S(R(x))$ for this case, we first extend a ray from $(0, 0)$ through $(x_T, f(x_T))$ and out some distance δ where it intersects $g(x)$. Note that

$$\delta > 0 \Rightarrow x_\delta < x_T < x_\epsilon$$

In order to minimize $S(g(x))$ under these conditions, define

$$g_0(x) = \begin{cases} y = \dfrac{g(x_\delta) - y_A}{x_\delta - x_A}(x - x_A) + y_A & if x \in [x_A, x_\delta \\ y = \dfrac{g(x_\epsilon) - g(x_\delta)}{x_\epsilon - x_\delta}(x - x_\epsilon) + g(x_\epsilon) & if x \in [x_\delta, x_\epsilon] \end{cases}$$

Therefore, $g_0(x)$ takes a straight line path from A to the end of the extended ray δ and then a straight line from δ to $f(x_\epsilon)$. Note that since $\delta > 0$, $S(g_0(x)) \leq S(g(x))$. Now, we consider $S(g_0(x))$ on the following intervals: $x \in [x_A, x_\delta]$ and $x \in [x_\delta, x_\epsilon]$ and compare to $S(R(x))$ on $x \in [x_A, x_T]$ and $x \in [x_T, x_\epsilon]$ respectively.

Since δ is an extension of the radius, r, of $f(x)$, $g_0(x)$ for $x \in [x_A, x_\delta]$ is the hypothenuse of the right triangle formed by $g_0(x)$, $R(x)$, and δ. Thus, by the Pythagorean Theorem, comparing $S(g_0(x))$ and $S(R(x))$ on their respective first intervals mentioned above will always yield

$$S(g(x)) \geq S(g_0(x)) > S(R(x)) \tag{38}$$

Now, with concern to the remaining intervals, $g(x) : x \in [x_\delta, x_\epsilon]$ and $R(x) : x \in [x_T, x_\epsilon]$, we simply need to show that $S(g(x)) \geq S(R(x))$ in order for $R(x)$ to be the optimized path along the entire interval $[x_A, x_M]$.

In order to minimize $S(g(x)) : x \in [x_\delta, x_\epsilon]$ we take the limit of $S(g(x))$ as δ approaches 0. Clearly,

$$\delta \to 0 \Rightarrow g(x) \to R(x) = f(x) : x \in [x_\delta, x_\epsilon] \tag{39}$$

due to the boundary condition imposed on $g(x)$ and $R(x)$. Thus

$$\lim_{\delta \to 0} x_\delta = x_T \Rightarrow \lim_{\delta \to 0} S(g(x)) = S(R(x)) \tag{40}$$

Therefore,

$$S(R(x)) \leq S(g(x)) \forall x \in [x_A, x_\epsilon] \tag{41}$$

Note that since for this case, $R(x) = g(x) \forall x \in [x_\epsilon, x_M]$, this above inequality regarding the arc lengths on the domain $[x_A, x_\epsilon]$ is sufficient for the entire case. Therefore,

$$S(R(x)) < S(g(x)) : x \in [x_A, x_M] \tag{42}$$

This completes the proof.

Time Is Of The Essence

With more time to consider applications of this process, the original inspiration of a roadway would be a lengthy and satisfying application of the aforementioned procedure. Using a set of points that would define the perimeter of the road, a boundary function can be created using a curve fitting technique such as splining. By shifting the function that defines one side of the road a number of units corresponding to the number of lanes, the two functions create upper and lower bounds for the optimum path. From there, the region can be split into reasonable subregions where the boundary functions have the right characteristics to create obstacle circles for the application of our process. Creating the optimum path in each region and fitting those curves together can make a reasonable approximation for the overall optimum path a driver should take along the multilane road from their start position to the desired destination.

Furthermore, analytic solutions to this problem for actual elastic bands would also be a worthy application for any physics, math, or engineering student. Working the optimum path problem from an energy standpoint is a significant change of perspective that should lead to a supporting solution. In doing so, attacking the task using calculus of variations and Lagrangian/Hamiltonian might allow for a more elegant and seemingly powerful method to devise optimum paths for physical circumstances. If analogous connections can then be made between the physical approach and the general approach, it may be the case that a separate, more fluid and supporting process can be used to find optimum paths in two-space.

More directly related to this project, an extension using a quadratic boundary arc coupled with a second-order Taylor Series expansion for generalization, one could create the Rubber Band Solution for a different set of boundary functions. Once the solution was created, the Taylor Series approximation could be used to approximate the Rubber Band Solution for a more general set of boundary functions. Comparing these approximate optimized paths to those created using a radius of curvature technique lends itself to an error analysis study between the two approximation methods.

Ultimately, a complete generalization of the problem is ideal. If a process can be made to find the optimum path through any bounded region using any set of functions as boundaries, the process of using circles as obstacles becomes obsolete for the more general solution. However, the accessibility to this level of generalization seems to be outside of the scope of undergraduate studies. It is also natural to desire an extension to three dimensions for optimization of paths between two endpoints. This is an open area of research in mathematics and is therefore best left to the professionals with the tools to attack the problem.

Appendix A

Here we will demonstrate the process mathematically for the most general case using an obstacle circle in two-space and a path underneath unlike the path illustrated previously.

Given an obstacle circle with center coordinates (x_0, y_0) and radius r and endpoints $A = (x_A, y_A)$ and $B = (x_B, y_B)$

$$(x - x_0)^2 - (y - y_0)^2 = r^2 \tag{43}$$

We first translate the boundary arc to the origin

$$x^2 + y^2 = r^2 \tag{44}$$

and the endpoints about it so that

$$A = (x_A - x_0, y_A - y_0) = (x_{AT}, y_{AT}) \tag{45}$$

$$B = (x_B - x_0, y_B - y_0) = (x_{BT}, y_{BT}) \tag{46}$$

Note that it would be the next step in the process to create the line of symmetry and test against it in order to decide which path needed to be taken. Since we have preselected to illustrate a path underneath the boundary arc, the line of symmetry and condition is as follows:

$$\begin{aligned} y_S - (y_A - y_0) &= \frac{y_A - y_0}{x_A - x_0}[x_S - (x_A - x_0)] \\ \Rightarrow y_S - y_{AT} &= \frac{y_{AT}}{x_{AT}}(x_S - x_{AT}) \\ \Rightarrow y_S &= \frac{y_{AT}}{x_{AT}}(x_S - x_{AT}) + y_{AT} \end{aligned} \tag{47}$$

Thus, in order for a path underneath to be the optimum path,

$$y_{SB} = \frac{y_{AT}}{x_{AT}}(x_B - x_{AT}) + y_{AT} > y_{BT}$$

Now, like before, the system should be rotated so that point A lies on the horizontal axis.

$$A_{TR} = (x_{ATR}, y_{ATR}) = \left(-\sqrt{x_{AT}^2 + y_{AT}^2}, 0\right) \tag{48}$$

yielding the same tangential point as proven in Theorem 1. However, due to the needed negative y value for a path underneath the obstacle arc and the symmetry of the system relative to point A about the x-axis, the tangential point is as follows:

$$(x_{ATRI}, y_{ATRI}) = \left(\frac{-r^2}{R_A}, -r\sqrt{1 - \left(\frac{r}{R_A}\right)^2}\right) \tag{49}$$

Notice that this is the tangential point to the obstacle arc when it is still in its translated and rotated state. In other words, the system is centered at the origin and endpoint A still lies on the x-axis. In order to find the actual tangential point for the system in its original state, this given representation must be rotated and translated by the same amount that endpoint A is rotated and translated to return it to its original position in space.

First we rotate the system back in its translated position. This yields a tangential point, (x_{ATI}, y_{ATI}), corresponding to endpoint A such that

$$(x_{ATI}, y_{ATI}) = (r\cos\theta, r\sin\theta) \tag{50}$$

where the value of θ from Theorem 3 holds in this case as well.

$$\theta = \pi + \tan^{-1}\left(\frac{y_{AT}}{x_{AT}}\right) + \tan^{-1}\left(\frac{y_{ATRI}}{x_{ATRI}}\right) \tag{51}$$

At last, the system can be translated back to its original position returning the original equation for the boundary arc, values for endpoint A, and the value

$$(x_{AI}, y_{AI}) = (r\cos\theta - x_0, r\sin\theta - y_0) \tag{52}$$

with θ defined in the same way it was in the translated state.

These complete the set of values necessary to create the optimum path from endpoint A to the tangential point on the boundary arc for a path along the bottom of the obstacle.

The calculations and configurations for the B section of the path are identical to those for the A section excluding the value for θ in the representation of the tangential point. The correct values are as follows:

$$(x_{BTRI}, y_{BTRI}) = \left(\frac{r^2}{R_B}, -r\sqrt{1-\left(\frac{r}{R_B}\right)^2}\right) \tag{53}$$

is rotated so that

$$(x_{BTI}, y_{BTI}) = (r\cos\theta, r\sin\theta) \tag{54}$$

defining θ so that

$$\theta = \tan^{-1}\left(\frac{y_{BT}}{x_{BT}}\right) + \tan^{-1}\left(\frac{y_{BTRI}}{x_{BTRI}}\right) \tag{55}$$

and then finally translating the system back to its original position to finalize the tangential point so that

$$(x_{BI}, y_{BI}) = (r\cos\theta, r\sin\theta) \tag{56}$$

with θ retaining the value above for the B section of the path.

The complete Rubber Band Solution is identical to the one on display in Theorem 4. The path direction does not effect the way that the solution is defined, merely the values that fill the linear equations present in the solution.

Appendix B

Here we will show a full example using a real-valued function in two-space and two endpoints. The calculations and formulation of the Rubber Band Solution will be made for the A and B segments simultaneously.

$$f(x) = \frac{-x^2}{2} \tag{57}$$

$$A = \left(-2, \frac{-3}{2}\right), B = \left(4, \frac{-1}{2}\right) \tag{58}$$

Now we must create the obstacle circle as an approximation for $f(x)$. Beginning, we investigate the first derivate of $f(x)$ on $[-2, 4]$.

$$f'(x) = -x \tag{59}$$

then, we see that the critical point is as follows:

$$\begin{aligned} f'(x) &= -x_c = 0 \\ \Rightarrow x_c &= 0 \end{aligned} \tag{60}$$

thus the relative extrema value of $f(x)$ is indeed in our interval of concern and the value is

$$(x_c, y_c) = (x_c, f(x_c)) = (0, f(0)) = (0, 0) \tag{61}$$

Next we want to solve for the radius of curvature at this point in order to construct the obstacle circle. Therefore,

$$\begin{aligned} R_{f(x_c)} &= \frac{[1 + f'(x_c)^2]^{3/2}}{f''(x_c)} \\ \Rightarrow R_{f(0)} &= \frac{[1 + f'(0)^2]^{3/2}}{f''(0)} \\ \Rightarrow R_{f(0)} &= \frac{1^{3/2}}{-1} \\ \Rightarrow R_{f(0)} &= -1 \end{aligned} \tag{62}$$

It follows that

$$(x_0, y_0) = (x_c, f(x_c) + R_{f(x_c)}) = (0, -1) \tag{63}$$

and

$$\begin{aligned} (x - x_0)^2 + (y - y_0)^2 &= r^2 \\ \Rightarrow (x - x_0)^2 + (y - y_0)^2 &= R_{f(x_c)}^2 \\ \Rightarrow (x - 0)^2 + (y + 1)^2 &= 1^2 \\ \Rightarrow x^2 + (y + 1)^2 &= 1 \end{aligned} \tag{64}$$

Now that we have a system representation with an obstacle circle and two endpoints, we can implement the steps laid out in the general solution. Therefore, we translate the entire system about the origin yielding

$$x^2 + y^2 = 1 \tag{65}$$

$$A_T = (x_{AT}, y_{AT}) = \left(-2, -\frac{1}{2}\right), B_T = (x_{BT}, y_{BT}) = \left(4, \frac{1}{2}\right) \tag{66}$$

Now it is most helpful to look at the line of symmetry condition for the system in this translated state in order to find whether an over the top or underneath path will yield the true optimized path around the obstacle circle.

Therefore, we have

$$\begin{aligned} y - y_0 &= \frac{y_{AT} - y_0}{x_{AT} - x_0}(x - x_0) \\ \Rightarrow y - 0 &= \frac{-\frac{1}{2} - 0}{-2 - 0}(x - 0) \\ \Rightarrow y &= \frac{1}{4}x \end{aligned} \tag{67}$$

Then testing the line of symmetry against point B,

$$y(x_{BT}) = \frac{1}{4}(4) = 1 > \frac{1}{2} \tag{68}$$

Therefore, we see that a path underneath the obstacle circle is desired for optimization.

Since the system is in a translated state, the system must be rotated so that A and B lie on the x-axis. Note that although these calculations are simultaneously, it is proper to think of the entire system rotating in a single direction, not in opposite directions at the same time. Thus,

$$A_{TR} = \left(-\sqrt{x_{AT}^2 + y_{AT}^2}, 0\right) = \left(-\sqrt{\frac{17}{4}}, 0\right) \tag{69}$$

$$B_{TR} = \left(\sqrt{x_{BT}^2 + y_{BT}^2}, 0\right) = \left(\sqrt{\frac{65}{4}}, 0\right) \tag{70}$$

yielding

$$(x_{AITR}, y_{AITR}) = \left(\frac{-r^2}{R_A}, -r\sqrt{1 - \left(\frac{r}{R_A}\right)^2}\right) = \left(\frac{2}{\sqrt{17}}, -\sqrt{\frac{13}{17}}\right) \tag{71}$$

$$(x_{BITR}, y_{BITR}) = \left(\frac{r^2}{R_B}, -r\sqrt{1 - \left(\frac{r}{R_B}\right)^2}\right) = \left(\frac{2}{\sqrt{65}}, -\sqrt{\frac{61}{65}}\right) \tag{72}$$

Now that we have the tangential points defined in the translated and rotated state of the system, we define them using coordinates relative to the original position of the system.

$$(x_{AI}, y_{AI}) = (r\cos\theta - x_0, r\sin\theta - y_0) \tag{73}$$

Looking at each component,

$$\begin{aligned}
x_{AI} &= \cos\left[\pi + \tan^{-1}\left(\frac{-\frac{1}{2}}{-2}\right) + \tan^{-1}\left(\frac{-\sqrt{\frac{13}{17}}}{-\frac{2}{\sqrt{17}}}\right)\right] - 0 \\
&= \cos\left[\pi + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{\sqrt{13}}{2}\right)\right] \\
&= -0.258497
\end{aligned}$$

$$\begin{aligned}
y_{AI} &= \sin\left[\pi + \tan^{-1}\left(\frac{-\frac{1}{2}}{-2}\right) + \tan^{-1}\left(\frac{-\sqrt{\frac{13}{17}}}{-\frac{2}{\sqrt{17}}}\right)\right] - 1 \\
&= \sin\left[\pi + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{\sqrt{13}}{2}\right)\right] - 1 \\
&= -1.966012
\end{aligned}$$

Then similarly, we have

$$(x_{BI}, y_{BI}) = (r\cos\phi - x_0, r\sin\phi - y_0) \tag{74}$$

Looking at each component,

$$\begin{aligned}
x_{BI} &= \cos\left[\tan^{-1}\left(\frac{\frac{1}{2}}{4}\right) + \tan^{-1}\left(\frac{-\sqrt{\frac{61}{65}}}{\frac{2}{\sqrt{65}}}\right)\right] - 0 \\
&= \cos\left[\tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(-\frac{\sqrt{61}}{2}\right)\right] \\
&= 0.366311
\end{aligned}$$

$$
\begin{aligned}
y_{BI} &= \sin\left[\tan^{-1}\left(\frac{\frac{1}{2}}{4}\right) + \tan^{-1}\left(\frac{-\sqrt{\frac{61}{65}}}{\frac{2}{\sqrt{65}}}\right)\right] - 1 \\
&= \sin\left[\tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(-\frac{\sqrt{61}}{2}\right)\right] - 1 \\
&= -1.930492
\end{aligned}
$$

Now that we have every needed point value defined for the system about its original position, we can define the Rubber Band Solution for this system as follows:

$$
R(x) = \begin{cases}
y = \dfrac{(-1.966012 + 1.5)}{(-0.258497 + 2)}(x+2) - 1.5 & \text{if } x \in [-2, -0.258497] \\
y = -\sqrt{1-x^2} - 1 & \text{if } x \in [-0.258497, 0.366311] \\
y = \dfrac{(-1.930492 + 0.5)}{(0.366311 - 4)}(x-4) - 0.5 & \text{if } x \in [0.366311, 4]
\end{cases},
$$

that is defined more clearly as:

$$
R(x) = \begin{cases}
y = -0.267597x - 2.035194 & \text{if } x \in [-2, -0.258497] \\
y = -\sqrt{1-x^2} - 1 & \text{if } x \in [-0.258497, 0.366311] \\
y = 0.393675x - 2.074699 & \text{if } x \in [0.366311, 4]
\end{cases},
$$

Appendix C

Below is a sample MATLAB m.file that can be used to complete the path optimization process for a circle obstacle. Note that although the code has all of the components of the generalized version, due to the defined calculation of $tan^{-1}(x)$, working on some intervals in the real plane will not yield precise solutions for the tangent points. It is best if this algorithm is used for circles and endpoints relatively close to the origin.

```
function RubberBandSolution(xa,ya,xb,yb,x0,y0,r)
% RubberBandSolution creates the Rubber
% Band Solution for a given set of points and obstacle circle
% Inputs:
%   (xa,ya) = x and y coordinates for the endpoint A
%   (xb,yb) = x and y coordinates for the endpoint B
%   (x0,y0) = x and y coordinates for the center of the obstacle circle
%   r = radius of the obstacle circle

if nargin<5,error('five arguments needed: (xa,ya,xb,yb,r)'),end;
if nargin<6, y0=0; x0=0;end
if sqrt(xa^2 + ya^2)<=r,error('endpoint must be outside of the
 obstacle function'),end
if sqrt(xb^2 + yb^2)<=r,error('endpoint must be outside of the
 obstacle function'),end

plot(xa,ya,'ok','markersize',8)
hold on
plot(xb,yb,'ok','markersize',8)
hold on

theta=0:pi/100:2*pi;
plot(r*cos(theta)+x0,r*sin(theta)+y0,'k','linewidth',2)
hold on

xo=r*cos(theta); % x value of the obstacle circle centered at 0
yo=r*sin(theta); % y value of the obstacle circle centered at 0

xat=xa-x0; % translated/shifted values for the endpoints
yat=ya-y0;
xbt=xb-x0;
ybt=yb-y0;

xatr=-sqrt(yat^2+xat^2); % translated and rotated values for
 the endpoints. they now lie on the x-axis
yatr=0;
xbtr=sqrt(ybt^2+xbt^2);
```

```
ybtr=0;

RA=-xatr; % Radii for the above circles through each endpoint
RB=xbtr;

xA=RA*cos(theta); % x and y coordinates for the circles through
 the endpoints centered at 0
yA=RA*sin(theta);
xB=RB*cos(theta);
yB=RB*sin(theta);

ys=(yat/xat)*(xbt-xat)+yat;

if ybt>=ys, %The path over the top half will be the desired route
    xaitr=-r^2/RA;
    yaitr=r*sqrt(1-(r/RA)^2);
    xai=r*cos((180-atan(yaitr/xaitr)+atan((yat/xat))))+x0;
    yai=r*sin((180-atan(yaitr/xaitr)+atan((yat/xat))))+y0;
    xbitr=r^2/RB;
    ybitr=r*sqrt(1-(r/RB)^2);
    xbi=r*cos(atan(ybitr/xbitr)+atan((ybt/xbt)))+x0;
    ybi=r*sin(atan(ybitr/xbitr)+atan((ybt/xbt)))+y0;
else
    xaitr=-r^2/RA;
    yaitr=-r*sqrt(1-(r/RA)^2;
    xai = r*cos(180+atan(-r*sqrt(1-(r/RA)^2)/(-r^2/RA))
          +atan((yat/xat)));
    yai = r*sin(180+atan(-r*sqrt(1-(r/RA)^2)/(-r^2/RA))
          +atan((yat/xat)));
    xbitr=r^2/RB;
    ybitr=-r*sqrt(1-(r/RB)^2);
    xbi=r*cos(atan(ybitr/xbitr)-atan((ybt/xbt)))+x0;
    ybi=r*sin(atan(ybitr/xbitr)-atan((ybt/xbt)))+y0;
end

x=xa:(xb-xa)/100:xb;
y1=((yai-ya)/(xai-xa))*(x-xa)+ya;
y2=((ybi-yb)/(xbi-xb))*(x-xb)+yb;
plot(x,y1,'b','linewidth',2)
hold on
plot(x,y2,'m','linewidth',2)
hold on

end
```

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